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COMMENT

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Abstract. In a recent paper Oh and Singh determined a Hopf structure for a generalized q-oscillator algebra. We prove that under some general assumptions, the latter is, apart from some algebras isomorphic to $su_q(2)$, $su_q(1,1)$, or their undeformed counterparts, the only generalized deformed oscillator algebra that supports a Hopf structure. We show in addition that the latter can be equipped with a universal $\mathcal{R}$-matrix, thereby making it into a quasitriangular Hopf algebra.

In a recent paper (henceforth referred to as I and whose equations will be quoted by their number preceded by I), Oh and Singh [1] studied the relationships among various forms of the q-oscillator algebra and considered the conditions under which it supports a Hopf structure. They also presented a generalization of this algebra, together with its corresponding Hopf structure.

In the present comment, our purpose will be twofold. First, we plan to show that under some general assumptions about the coalgebra structure and the antipode map, the generalized q-oscillator algebra considered by Oh and Singh is, apart from some algebras isomorphic to $su_q(2)$, $su_q(1,1)$, or their undeformed counterparts, the only generalized deformed oscillator algebra (GDOA) that supports a Hopf structure. Second, we shall provide the universal $\mathcal{R}$-matrix for this deformed algebra and prove that the corresponding Hopf algebra is quasitriangular.

Let us introduce GDOAs as follows

Definition. Let $\mathcal{A}(G(N))$ be the associative algebra generated by the operators $\{1, a, a^\dagger, N\}$ and the function $G(N)$, satisfying the commutation relations

$$[N, a^\dagger] = a^\dagger \qquad [N, a] = -a \qquad [a, a^\dagger] = G(N) \tag{1}$$

and the Hermiticity conditions

$$(a)^\dagger = a^\dagger \qquad (a^\dagger)^\dagger = a \qquad N^\dagger = N \qquad (G(N))^\dagger = G(N) \tag{2}$$

where $G(z)$ is assumed to be an analytic function, which does not vanish identically.

For

$$G(N) = [\alpha N + \beta + 1]_q - [\alpha N + \beta]_q = \frac{\cosh(\varepsilon(\alpha N + \beta + 1/2))}{\cosh(\varepsilon/2)} \tag{3}$$

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where α and β are some real parameters, and $q = \exp \varepsilon \in \mathbb{R}^+$, $\mathcal{A}(G(N))$ reduces to the generalization of the q-oscillator algebra considered by Oh and Singh [1]†.

Note that the definition of $\mathcal{A}(G(N))$ differs from the usual definition of GDOAs [2], wherein both a commutation and an anticommutation relation

$$[a, a^\dagger] = F(N+1) - F(N) \qquad \{a, a^\dagger\} = F(N+1) + F(N) \qquad (4)$$

are imposed in terms of some structure function $F(z)$, assumed to be an analytic function, positive on some interval $[0, a)$ (where $a \in \mathbb{R}^+$ may be finite or infinite), and such that $F(0) = 0$. As in I, the reason for considering only the first relation in (4) is that the two relations do not prove compatible with a coalgebra structure.

As a consequence of its definition, the algebra $\mathcal{A}(G(N))$ has a Casimir operator defined by $C = F(N) - a^\dagger a$, where $F(N)$ is the solution of the difference equation $F(N+1) - F(N) = G(N)$, such that $F(0) = 0$. The present definition of GDOAs is therefore equivalent to the usual one [2] only in the representation wherein $C = 0$, i.e. in a Fock-type representation.

Let us now try to endow some of the algebras $\mathcal{A}(G(N))$ with a coalgebra structure and an antipode map, making them into Hopf algebras $\mathcal{H}$. For the coproduct, counit and antipode, let us postulate the following expressions:

$$\Delta(a^\dagger) = a^\dagger \otimes c_1(N) + c_2(N) \otimes a^\dagger \qquad \Delta(a) = a \otimes c_3(N) + c_4(N) \otimes a \qquad (5)$$

$$\Delta(N) = c_5 N \otimes 1 + c_6 1 \otimes N + \gamma 1 \otimes 1 \qquad (6)$$

$$\epsilon(a^\dagger) = c_7 \qquad \epsilon(a) = c_8 \qquad \epsilon(N) = c_9 \qquad (7)$$

$$S(a^\dagger) = -c_{10}(N)a^\dagger \qquad S(a) = -c_{11}(N)a \qquad S(N) = -c_{12}N + c_{13}1 \qquad (8)$$

where $c_i(N)$, $i = 1, \ldots, 4, 10, 11$, are functions of N, and c_i, $i = 5, \ldots, 9, 12, 13$, and γ are constants to be determined. Such expressions generalize those found in I for $G(N)$ given by (3), which correspond to

$$c_1(N) = (c_2(N))^{-1} = c_3(N) = (c_4(N))^{-1} = q^{\alpha(N+\gamma)/2}$$

$$c_5 = c_6 = c_{12} = 1 \qquad c_7 = c_8 = 0$$

$$c_9 = \tfrac{1}{2}c_{13} = -\gamma \qquad c_{10}(N) = (c_{11}(N))^{-1} = q^{\alpha/2} \qquad (9)$$

$$\gamma = \frac{2\beta + 1}{2\alpha} - i\frac{(2k+1)\pi}{2\alpha\varepsilon} \qquad k \in \mathbb{Z}.$$

To remain as general as possible, we shall not start by making any specific assumption about $G(N)$, except that it satisfies equations (1) and (2). For the moment, we shall also disregard the Hermiticity conditions (2) and work with complex algebras. Only at the end will conditions (2) be imposed.

In order that equations (5)–(8) define a Hopf structure, the so-far undetermined functions and parameters must be chosen in such a way that Δ, ϵ, and S satisfy the coassociativity, counit and antipode axioms, given in (I35), and that in addition, Δ and ϵ be algebra homomorphisms.

In accordance with equation (9), we shall start by assuming that in equation (6), γ takes a non-vanishing value. By substituting equation (6) into the coassociativity axiom (I35a), and taking into account that Δ must be an algebra homomorphism, we directly obtain

$$c_5 = c_6 = 1. \qquad (10)$$

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† Actually, Oh and Singh considered a slightly more general algebra, wherein the first two relations in (1) are also deformed by the introduction of a real parameter η. We shall not do so here, as this additional parameter can be incorporated into the definition of N by renormalizing the latter.

To derive the corresponding conditions for $a^\dagger$ and a, it is useful to expand the functions $c_i(N)$, $i = 1, \ldots, 4$, of equation (5) into power series

$$c_i(N) = \sum_{A=0}^{\infty} \frac{1}{A!} c_i^{(A)}(0) N^A \tag{11}$$

where $c_i^{(A)}(N)$ denotes the Ath derivative of $c_i(N)$, and to apply the relation

$$\Delta c_i(N) = \sum_{A,B=0}^{\infty} \frac{1}{A! \, B!} c_i^{(A+B)}(\gamma) N^A \otimes N^B$$

$$\text{if } \Delta(N) = N \otimes 1 + 1 \otimes N + \gamma 1 \otimes 1. \tag{12}$$

We then obtain in a straightforward way that $c_i(N)$, $i = 1, \ldots, 4$, must satisfy the equations

$$c_i^{(A)}(0) c_i^{(B)}(0) = c_i^{(A+B)}(\gamma) \qquad i = 1, \ldots, 4 \qquad A, B = 0, 1, 2, \ldots. \tag{13}$$

By substituting now equations (5)–(8) into the counit and antipode axioms (I35b) and (I35c), we easily get

$$c_i(-\gamma) = 1 \qquad i = 1, \ldots, 4 \tag{14}$$

$$c_7 = c_8 = 0 \qquad c_9 = -\gamma \tag{15}$$

and

$$c_1(-N + 1 - 2\gamma) = c_2(N)\, c_{10}(N) \qquad c_2(-N - 2\gamma) = c_1(N - 1)\, c_{10}(N) \tag{16}$$

$$c_3(-N - 1 - 2\gamma) = c_4(N)\, c_{11}(N) \qquad c_4(-N - 2\gamma) = c_3(N + 1)\, c_{11}(N) \tag{17}$$

$$c_{12} = 1 \qquad c_{13} = -2\gamma \tag{18}$$

respectively.

It remains to impose that the algebra and coalgebra structures are compatible. By applying Δ or ϵ to both sides of the first two equations contained in (1), we obtain identities, while by doing the same with the third one and using equations similar to (11) and (12) for $G(N)$, we are led to the conditions

$$c_2(N + 1) \otimes c_3(N) = c_2(N) \otimes c_3(N - 1)$$
$$c_4(N) \otimes c_1(N + 1) = c_4(N - 1) \otimes c_1(N) \tag{19}$$

$$G^{(A-B)}(0)(c_1 c_3)^{(B)}(0) + (c_2 c_4)^{(A-B)}(0) G^{(B)}(0) = G^{(A)}(\gamma)$$

$$A = 0, 1, 2, \ldots \qquad B = 0, 1, \ldots A \tag{20}$$

and

$$G(-\gamma) = 0. \tag{21}$$

We note that the Hopf axioms directly fix the values of all the constants c_i, $i = 5, \ldots, 9, 12, 13$, in terms of the remaining one γ, but that the seven functions $c_i(N)$, $i = 1, \ldots, 4, 10, 11$, and $G(N)$ are only implicitly determined by equations (13), (14), (16), (17), and (19)–(21). We shall now proceed to show that the latter can be solved to provide explicit expressions for the yet unknown functions of N in terms of γ and of some additional parameters.

Considering first the two conditions in (19), we immediately see that they can only be satisfied if there exist some complex constants k_1, k_2, such that

$$c_1(N + 1) = k_1 c_1(N) \qquad c_4(N) = k_1^{-1} c_4(N - 1)$$
$$c_2(N + 1) = k_2 c_2(N) \qquad c_3(N) = k_2^{-1} c_3(N - 1). \tag{22}$$

These relations in turn imply that

$$c_1(N) = \alpha_1 e^{\kappa_1 N} \qquad c_2(N) = \alpha_2 e^{\kappa_2 N} \qquad c_3(N) = \alpha_3 e^{-\kappa_2 N} \qquad c_4(N) = \alpha_4 e^{-\kappa_1 N} \qquad (23)$$

where $\kappa_1 = \ln k_1$, $\kappa_2 = \ln k_2$, and α_i, $i = 1, \ldots, 4$, are some complex parameters. The latter are determined by condition (14) as

$$\alpha_1 = e^{\kappa_1 \gamma} \qquad \alpha_2 = e^{\kappa_2 \gamma} \qquad \alpha_3 = e^{-\kappa_2 \gamma} \qquad \alpha_4 = e^{-\kappa_1 \gamma}. \qquad (24)$$

It is then straightforward to check that the functions $c_i(N)$, $i = 1, \ldots, 4$, defined by (23) and (24), automatically satisfy condition (13).

By inserting now equations (23) and (24) into conditions (16) and (17), we directly obtain the following explicit expressions for $c_{10}(N)$ and $c_{11}(N)$:

$$c_{10}(N) = e^{-(\kappa_1 + \kappa_2)(N + \gamma) + \kappa_1} \qquad c_{11}(N) = e^{(\kappa_1 + \kappa_2)(N + \gamma) + \kappa_2}. \qquad (25)$$

The same substitution performed in condition (20) transforms the latter into

$$(\kappa_1 - \kappa_2)^B e^{(\kappa_1 - \kappa_2)\gamma} G^{(A-B)}(0) + (-1)^{A-B} (\kappa_1 - \kappa_2)^{A-B} e^{-(\kappa_1 - \kappa_2)\gamma} G^{(B)}(0) = G^{(A)}(\gamma)$$
$$A = 0, 1, 2, \ldots \qquad B = 0, 1, \ldots, A. \qquad (26)$$

It can be easily shown by induction over A that whenever $\kappa_1 \neq \kappa_2$, the solution of recursion relation (26) is given by

$$G^{(A)}(0) = \begin{cases} (\kappa_1 - \kappa_2)^A \, G(0) & \text{if } A \text{ is even} \\ (\kappa_1 - \kappa_2)^A \coth\big((\kappa_1 - \kappa_2)\gamma\big) G(0) & \text{if } A \text{ is odd} \end{cases} \qquad (27)$$

and

$$G^{(A)}(\gamma) = \begin{cases} (\kappa_1 - \kappa_2)^A \, G(\gamma) & \text{if } A \text{ is even} \\ (\kappa_1 - \kappa_2)^A \coth\big(2(\kappa_1 - \kappa_2)\gamma\big) G(\gamma) & \text{if } A \text{ is odd} \end{cases} \qquad (28)$$

where

$$G(\gamma) = 2 \cosh\big((\kappa_1 - \kappa_2)\gamma\big) G(0). \qquad (29)$$

From equation (27) and the Taylor expansion of $G(N)$, we then obtain

$$G(N) = G(0) \frac{\sinh\big((\kappa_1 - \kappa_2)(N + \gamma)\big)}{\sinh\big((\kappa_1 - \kappa_2)\gamma\big)} \qquad \kappa_1 \neq \kappa_2. \qquad (30)$$

Such a function also satisfies (28) and (29), as well as the remaining condition (21). Equations (27)–(30) remain valid for $\kappa_1 = \kappa_2$ provided appropriate limits are taken. In such a case, function (30) becomes

$$G(N) = G(0) \left(1 + \frac{N}{\gamma}\right) \qquad \kappa_1 = \kappa_2. \qquad (31)$$

Had we taken $\gamma = 0$ instead of $\gamma \neq 0$ in (6), a similar analysis would have led to

$$G(N) = \begin{cases} G^{(1)}(0) \dfrac{\sinh\big((\kappa_1 - \kappa_2)N\big)}{\kappa_1 - \kappa_2} & \text{if } \kappa_1 \neq \kappa_2 \\ G^{(1)}(0) \, N & \text{if } \kappa_1 = \kappa_2 \end{cases} \qquad (32)$$

and a Hopf structure given by (10), (15), (18), and (23)–(25), but where γ is set equal to 0. For an appropriate choice of $G^{(1)}(0)$ (obtained by renormalizing $a^\dagger$ and a if necessary), such a form of $G(N)$ corresponds to the complex q-algebra $sl_q(2)$ if $\kappa_1 \neq \kappa_2$, and to $sl(2)$ if $\kappa_1 = \kappa_2$ [3].

The remaining step in the construction of algebras $\mathcal{A}\big(G(N)\big)$ with a Hopf structure consists in imposing the Hermiticity conditions (2) on the algebraic structure. They require

that the function $G(N)$, defined in (30), (31), or (32), be a real function of N. For the latter choice, we obtain the real forms of $sl_q(2)$ or $sl(2)$, namely $su_q(2)$ and $su_q(1,1)$, or $su(2)$ and $su(1,1)$ [3]. It remains to consider the former choices for γ non-real, since the real γ case comes down to the $\gamma = 0$ one by changing N into $N + \gamma$. For such γ values, function (31) cannot be Hermitian. It therefore only remains to consider the case where $G(N)$ is given by (30).

In such a case, the discussion of the hermiticity conditions is rather involved as $G(N)$ depends upon two complex parameters $\kappa_1 - \kappa_2$, and γ, in addition to the non-vanishing real parameter $G(0)$. By setting

$$\kappa_1 = \xi_1 + i\eta_1 \qquad \kappa_2 = \xi_2 + i\eta_2 \qquad \kappa \equiv \kappa_1 - \kappa_2 = \xi + i\eta \qquad \gamma = \gamma_1 + i\gamma_2 \qquad (33)$$

where $\xi_1, \eta_1, \xi_2, \eta_2, \xi, \eta, \gamma_1, \gamma_2 \in \mathbb{R}$, the function $G(N)$, defined in (30), can be rewritten as

$$G(N) = G(0) \left(\alpha(N) + i\beta(N) \right) \qquad (34)$$

where

$$
\begin{aligned}
&\alpha(N) = \frac{a(N)c + b(N)d}{c^2 + d^2} \qquad \beta(N) = \frac{b(N)c - a(N)d}{c^2 + d^2} \\
&a(N) = \sinh(A(N)) \cos(B(N)) \qquad b(N) = \cosh(A(N)) \sin(B(N)) \\
&c = \sinh C \cos D \qquad d = \cosh C \sin D \qquad (35) \\
&A(N) = \xi(N + \gamma_1) - \eta\gamma_2 \qquad B(N) = \xi\gamma_2 + \eta(N + \gamma_1) \\
&C = \xi\gamma_1 - \eta\gamma_2 \qquad D = \xi\gamma_2 + \eta\gamma_1 .
\end{aligned}
$$

Hence, $G(N)$ is a real function of N if and only if

$$\beta(N) = 0 . \qquad (36)$$

Note that from the expressions of $\alpha(N)$ and $\beta(N)$ given in (35), it is clear that the parameter values for which c and d simultaneously vanish should be discarded.

Condition (36) has now to be worked out by successively combining the cases where $\gamma_1 = 0$ and $\gamma_2 \neq 0$, or $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$, with those where $\xi \neq 0$ and $\eta = 0$, $\xi = 0$ and $\eta \neq 0$, or $\xi \neq 0$ and $\eta \neq 0$. For instance, if $\gamma_1, \gamma_2, \xi \neq 0$ and $\eta = 0$, equation (35) can be written as

$$
\begin{aligned}
\cosh\!\big(\xi(N + \gamma_1)\big) \sin(\xi\gamma_2) \sinh(\xi\gamma_1) \cos(\xi\gamma_2) \\
= \sinh\!\big(\xi(N + \gamma_1)\big) \cos(\xi\gamma_2) \cosh(\xi\gamma_1) \sin(\xi\gamma_2) .
\end{aligned} \qquad (37)
$$

As both sides of this relation have a different dependence on N, they must identically vanish. Since $\xi \neq 0$ by hypothesis, we must therefore have either $\sin(\xi\gamma_2) = 0$ or $\cos(\xi\gamma_2) = 0$. The first condition leads to $\gamma_2 = k\pi/\xi$, $k \in \mathbb{Z}_0$, while the second gives rise to $\gamma_2 = (2k + 1)\pi/(2\xi)$, $k \in \mathbb{Z}$.

Similarly, if we assume that $\gamma_1, \gamma_2, \eta \neq 0$, and $\xi = 0$, we obtain that equation (36) is equivalent to

$$\sin\!\big(\eta(N + \gamma_1)\big) \cos(\eta\gamma_1) = \cos\!\big(\eta(N + \gamma_1)\big) \sin(\eta\gamma_1) \qquad (38)$$

or, by using some trigonometric identities,

$$\sin(\eta N) = 0 . \qquad (39)$$

As $\eta \neq 0$, this relation cannot be satisfied as an operator identity.

By proceeding in this way, one can easily show the following result.

Proposition 1. The algebras $\mathcal{A}(G(N))$ that support a Hopf structure of type (5)–(8) and are not isomorphic to $su_q(2)$, $su_q(1,1)$, $su(2)$, $su(1,1)$, are determined by equations (10), (15), (18), (23)–(25) and the following conditions:

$$G(N) = G(0)\frac{\cosh(\xi(N+\gamma_1))}{\cosh(\xi\gamma_1)} \qquad G(0), \xi \in \mathbb{R}_0 \qquad \gamma_1 \in \mathbb{R}$$

$$\kappa_1 - \kappa_2 = \xi \qquad \gamma = \gamma_1 + \mathrm{i}\frac{(2k+1)\pi}{2\xi} \qquad k \in \mathbb{Z}.$$

(40)

Remark. The isomorphism referred to in the proposition is an algebra (not a Hopf algebra) isomorphism. One can indeed obtain algebras $\mathcal{A}(G(N))$ that have the commutation relations and Hermiticity conditions of $su_q(2)$, $su_q(1,1)$, $su(2)$, $su(1,1)$, but more general expressions for the coproduct, the counit and the antipode.

Comparing the results of proposition 1 with equations (3) and (9), we notice that provided we set $\kappa_1 = -\kappa_2$, the Hopf algebra so obtained does coincide with that derived by Oh and Singh, the relations between the two sets of parameters being given by

$$G(0) = \frac{\cosh(\varepsilon(2\beta+1)/2)}{\cosh(\varepsilon/2)} \qquad \xi = \alpha\varepsilon \qquad \gamma_1 = \frac{2\beta+1}{2\alpha}.$$

(41)

Hence, we have the following,

Corollary 2. The only algebras $\mathcal{A}(G(N))$ that support a Hopf structure of type (5)–(8) and are not isomorphic to $su_q(2)$, $su_q(1,1)$, $su(2)$, $su(1,1)$, are isomorphic to those considered in I.

Remarks.
(i) The Hopf algebra obtained here is slightly more general than that constructed by Oh and Singh, as it contains the additional parameter $\kappa_1 + \kappa_2$.
(ii) Some further generalizations of the coproduct given in equations (5) and (6), obtained by introducing additional functions of N, fail to provide new Hopf algebras.

Let us now turn ourselves to the second point of this comment, namely the construction of the universal $\mathcal{R}$-matrix for the Oh and Singh Hopf algebra.

By first omitting the Hermiticity conditions, one obtains the following result.

Lemma 3. The complex Hopf algebras $\mathcal{H}$, defined by equations (1), (5)–(8), (10), (15), (18), (23)–(25) and (30), can be made into quasitriangular Hopf algebras by considering the element $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$, given by

$$\mathcal{R} = X^{-2(N+\gamma 1)\otimes(N+\gamma 1)} \sum_{n=0}^{\infty} \frac{(1-X^2)^n}{[n]_X!} X^{-n(n-1)/2} Y^n \lambda^{-2n}$$

$$\times \left((XY)^{(N+\gamma 1)}a\right)^n \otimes \left((XY)^{-(N+\gamma 1)}a^{\dagger}\right)^n$$

(42)

where

$$X = \mathrm{e}^{(\kappa_1-\kappa_2)/2} \qquad Y = \mathrm{e}^{(\kappa_1+\kappa_2)/2} \qquad \lambda^2 = -G(0)\frac{\sinh((\kappa_1-\kappa_2)/2)}{\sinh((\kappa_1-\kappa_2)\gamma)}$$

$$[n]_X = \frac{X^n - X^{-n}}{X - X^{-1}} \qquad [n]_X! = [n]_X[n-1]_X\ldots[1]_X \qquad [0]_X! = 1.$$

(43)

Proof. By direct substitution, one finds that $\mathcal{R}$, defined by (42) and (43), satisfies the relations

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \qquad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$$

$$\tau \circ \Delta(h) = \mathcal{R}\Delta(h)\mathcal{R}^{-1}$$

(44)

where $\mathcal{R}_{12}$, $\mathcal{R}_{13}$, $\mathcal{R}_{23} \in \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, and for instance $\mathcal{R}_{12} = \mathcal{R} \otimes I$, while τ is the twist operator, $\tau(a \otimes b) = b \otimes a$. □

By introducing now the additional conditions (40) and $\kappa_1 + \kappa_2 = 0$, and changing to Oh and Singh's notations (41), we obtain the final result.

Proposition 4. The Oh and Singh Hopf algebra, defined by equations (1)–(3), (5)–(9), is quasitriangular, with the $\mathcal{R}$-matrix given by

$$\mathcal{R} = q^{-\frac{1}{a}\left[(\beta+\frac{1}{2})^2 - P\left(\frac{2k+1}{2\ln q}\pi\right)^2 + \mathrm{i}\frac{(2\beta+1)(2k+1)}{2\ln q}\pi\right]} q^{-\alpha N \otimes N} \times \left(q^{-(\beta+\frac{1}{2}+\mathrm{i}\frac{2k+1}{2\ln q}\pi)N} \otimes q^{-(\beta+\frac{1}{2}+\mathrm{i}\frac{2k+1}{2\ln q}\pi)N} \right)$$

$$\times \sum_{n=0}^{\infty} \frac{[\mathrm{i}(-1)^k(q^{1/2}+q^{-1/2})]^n}{[n]_{q^{\alpha/2}}!} q^{-\alpha n(n-3)/4}\left((q^{\alpha N/2}a)^n \otimes (q^{-\alpha N/2}a^\dagger)^n\right). \quad (45)$$

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